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The slow dripping of a viscous fluid

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The problem considered is the determination of the mass of the drops which break away when a viscous liquid drips slowly out of a narrow vertical tube. A simple one-dimensional theory of the unsteady extension of a viscous thread under its own weight is given, which holds when viscosity, capillarity and gravity are important but inertia is negligible. A comparison with experiment is given. There are several systematic errors, the most important of which are associated with detailed behaviour at the pipe exit where die-swell and wetting are difficult to assess. With due allowance for these errors, agreement is fairly good.

1. Introduction

Suppose a liquid runs out of the open end of a narrow tube pointing vertically downwards (figure 1). Several flow regimes can be readily distinguished and some have been extensively studied. For example, if the flow rate is large enough a reasonably steady jet will form which accelerates and narrows under gravity, and will in many cases break up into droplets. This capillary instability of jets has been the subject of hundreds of papers and the recent survey by Bogy (1979) of one aspect of the problem may be cited as an example.

On the other hand, if the flow rate is very small the fluid will form a slowly growing pendant droplet which can be regarded as being in static equilibrium at all times. When the volume reaches a critical value it becomes unstable, part of it breaks away, and the remainder quickly recovers to form a smaller static droplet. The stability of drops has been the subject of a survey article by Michael (1981) and, of the works cited there, the paper by Padday & Pitt (1973) is of some relevance here. As well as analysing the stability of the drop Padday & Pitt discuss briefly the problem of determining the volume that breaks away once the stability limit is passed. There is experimental evidence that the volume breaking away depends on the flow rate even when the flow rate is very small, but there is fair agreement between the various theories and the various experiments, extrapolated to zero flow, and for the systems considered Rayleigh's formula (Rayleigh 1899),

$$\Delta m = 3.8\frac{\gamma R_0}{g}, \tag{1}$$

is a reasonable estimate. (Here Δm is the droplet mass, γ is the surface tension, and R_0 is the tube radius).

The present paper is an attempt to predict droplet sizes in a flow regime that is in a sense intermediate between the two described above. The fluid is of large viscosity (the systems summarized by Padday & Pitt (1973) are of low-viscosity fluids) and the flow rate is too low to establish a jet. Instead, a drop forms and grows slowly into a short thread or column; this thread then stretches under its own weight and snaps

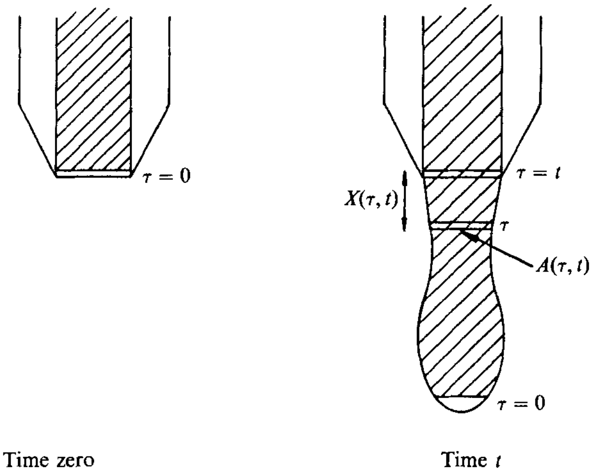

FIGURE 1. Sketch indicating the coordinate system.

off roughly half-way down. During this process, that is, up to the moment of rupture, the main balance of forces is between viscosity and gravity, with surface tension having an important but not dominant effect. Inertia is neglected throughout, though this involves a non-uniformity in the theory which is mentioned briefly below. After the thread ruptures, the upper part recoils fairly rapidly to form a new drop and the process repeats, so that a periodic dripping process will be seen.

In the theory to be presented a simple quasi-one-dimensional approach is used to consider the extension of an extruding viscous thread as it sags under its own weight. The approximations used will of course fail near the bottom end of the thread, and quite probably during the recoil process; but it turns out that the quantity of interest, namely the breakaway volume, is not affected by this. A short programme of experiments was carried out to test the theory. There are one major and two minor sources of error, that is, points at which the model departs from reality; and when the estimated corrections are made the agreement is fairly good.

2. Theory

In order to make the calculations as clear as possible and to explain the choice of dimensionless variables, we begin by neglecting surface tension and by analysing the motion from the instant when liquid first emerges from the tube up to the time when the first drop falls away. The estimate of the size of the first drop is of some interest in itself, and can be modified to include the effects of surface tension without much difficulty. We then turn to the periodic dripping motion which is established after a time.

A thread of liquid emerges from the tube and gradually extends downwards, stretching somewhat under its own weight. We assume that the velocity profile is uniform, that is, plug flow; the relaxation of the parabolic profile which was established in the tube is assumed to occur quickly and is outside the scope of the present one-dimensional theory. Nonetheless this transition region is important and will be discussed later.

A Lagrangian coordinate system is employed in which fluid particles are labelled by the time, τ, at which they emerged from the tube. Thus if t is the present time, we have $0 \leqslant \tau \leqslant t$, with $\tau = 0$ on the bottom end of the thread and $\tau = t$ on the fluid element that is emerging at the present instant (see figure 1).

Let $X(\tau, t)$ be the distance below the orifice of a typical particle P, labelled by τ, at time t, and let $A(\tau, t)$ be its cross-section area. We suppose that the volume flow rate, Q, is constant. Then considering two neighbouring elements τ and $\tau + \mathrm{d}\tau$, the equation of conservation of volume reads

$$Q\,\mathrm{d}\tau = -A\,\mathrm{d}X,$$

i.e.

$$A\frac{\partial X}{\partial \tau} = -Q. \tag{2}$$

Next we consider the force balance on the fluid between τ and $\tau + \mathrm{d}\tau$, denoting the longitudinal stress by $S(\tau, t)$. (The sign convention here is that positive S corresponds to tension.) We find

$$(SA)_{\tau} - (SA)_{\tau+\mathrm{d}\tau} - \rho g A\,\mathrm{d}X = 0. \tag{3}$$

Thus

$$\frac{\partial}{\partial \tau}(SA) = -\rho g A \frac{\partial X}{\partial \tau} = \rho g Q, \tag{4}$$

which can be integrated to give

$$SA = \rho g Q \tau \tag{5}$$

because $S = 0$ at $\tau = 0$. Of course this equation has the simple interpretation that the longitudinal force at P equals the weight of all the fluid below.

Finally we need a constitutive equation, and for a Newtonian liquid of viscosity μ this is

$$S = -3\mu \frac{1}{A}\frac{\partial A}{\partial t}. \tag{6}$$

(The factor 3μ is the usual Trouton result for elongational flow; see, for example, Petrie (1979). The strain rate for a fluid cylinder of length $L(t)$ is $\dot{L}/L$ which is the same as $-\dot{A}/A$.)

Equations (5) and (6) may now be combined to give

$$\frac{\partial A}{\partial t} = -\frac{\rho g Q \tau}{3\mu} \tag{7}$$

with the solution

$$A = A_0 - \frac{\rho g Q}{3\mu}(\tau t - \tau^2), \tag{8}$$

where A_0 is the tube cross-section area and the initial condition

$$A = A_0 \quad \text{at } \tau = t \tag{9}$$

has been used.

Thus the cross-section area of each element is reduced at a rate proportional to τ, and we want to discover which element shrinks to zero first (τ_c) and the time at which this occurs (t_c). This is easily found from (8) and we have

$$\tau_c = (3\mu A_0/\rho g Q)^{\frac{1}{2}}, \tag{10}$$

$$t_c = 2\tau_c. \tag{11}$$

It is interesting to note that exactly half the thread falls away as the first drop. What happens subsequently will be discussed later. It is possible to integrate (2) to obtain $X(\tau, t)$, and for the bottom of the thread $\tau = 0$ we have

$$X(0,t) = \frac{12\mu}{\rho g}\frac{1}{(t_c^2-t^2)^{\frac{1}{2}}}\arctan\frac{t}{(t_c^2-t^2)^{\frac{1}{2}}}. \tag{12}$$

This formula shows that the bottom of the thread goes to infinity in a finite time t_c, and so obviously the model must fail in some way before this. It can be shown that the singularity is removed when inertia is included in the equations, and that when this is done the cross-section area of the thread will tend to zero algebraically as $t \to \infty$. However, in the parameter regime of interest inertial effects become important only at times very close to t_c. The fact that the thread does break and the drop falls away is no doubt due to some mechanism of instability and this topic is discussed by White & Ide (1975); but for our purposes it is sufficient to note that the present theory applies up to times close to t_c since this is what determines the drop volume, however the rupture takes place. Some further remarks on this will be made later.

We now show how the theory can be modified to include surface tension. The force balance equation (5) is replaced by

$$SA + 2\pi R\gamma = \rho g Q\tau, \tag{13}$$

where R is the radius of the thread and γ is the surface tension. As before, this equates the total longitudinal force at P to the weight of the fluid below, and it can be seen that the effect of surface tension in modifying the shape of the thread at the bottom end is zero. Only the volume matters, not the detailed shape. It will still be necessary, however, to ensure that the fluid element at which the thread snaps is well within the one-dimensional region. The stress equation (6) is replaced by

$$S = -3\mu\frac{1}{A}\frac{\partial A}{\partial t} - \frac{\gamma}{R} \tag{14}$$

and the extra term represents a compressional stress γ/R. These equations can be combined to give a new equation for A, and it is convenient at this stage to introduce dimensionless variables. The time-scale for t and τ is $(\mu A_0/\rho g Q)^{\frac{1}{2}}$ and A is scaled on A_0. Using an asterisk to denote the dimensionless variables there results

$$\frac{\partial A^*}{\partial t^*} - \delta A^{*\frac{1}{2}} = -\tfrac{1}{3}\tau^*, \tag{15}$$

with the initial condition

$$A^* = 1 \quad \text{when } t^* = \tau^*. \tag{16}$$

Here δ is the dimensionless parameter

$$\delta = \gamma\left(\frac{\pi}{9\mu\rho g Q}\right)^{\frac{1}{2}}. \tag{17}$$

It is a simple matter to integrate (15) subject to (16) and we then put $A^* = 0$ to obtain a relation between the value of τ^* on any element and the time t^* at which it shrinks to zero cross-section. This is

$$t^* = \tau^* - \frac{2}{\delta}\left[1 + \frac{\tau^*}{3\delta}\log\frac{\tau^* - 3\delta}{\tau^*}\right]. \tag{18}$$

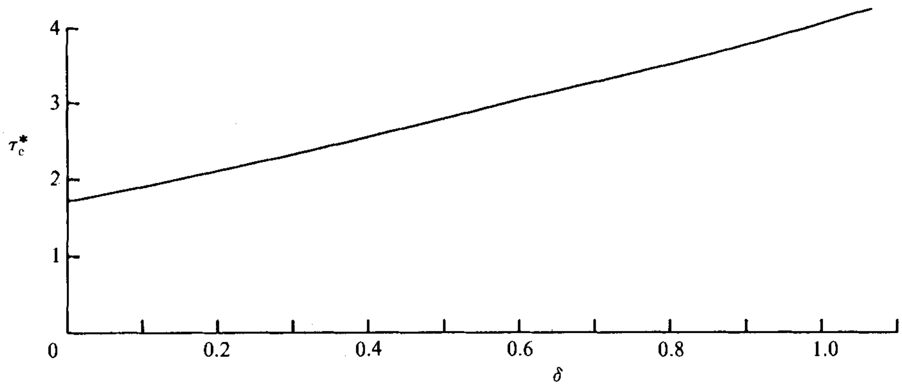

FIGURE 2. Graph of the dimensionless drop volume τ_c^* as a function of the surface-tension parameter δ (cf. equation (19)).

The element that reaches zero first satisfies an additional relation, namely $dt^*/d\tau^* = 0$. Thus the equation for τ_c^* is

$$\tau_c^{*2} = \frac{6}{\lambda}\left[\frac{1}{\lambda}\log(1-\lambda)+\frac{1}{1-\lambda}\right], \tag{19}$$

where

$$\lambda = 3\delta/\tau_c^*. \tag{20}$$

This implicit equation for τ_c^* can be easily solved for τ_c^* as a function of δ by first choosing λ, solving (19) for τ_c^* and then solving (20) for δ. The result is shown in figure 2.

The volume that breaks away is given essentially by τ_c^* and the time by t_c^*, which can be calculated from (18). This equation can be simplified, using (19), to give

$$t_c^* = \frac{6}{\tau_c^*(1-\lambda)} = \frac{6}{\tau_c^*-3\delta}. \tag{21}$$

Returning to dimensional variables, the formula for the breakaway mass Δm which follows from (21) is

$$\Delta m = \frac{6\mu A_0}{gt_c}+\frac{\pi\gamma R_0}{g}. \tag{22}$$

The theory gives a value of t_c, of course, in terms of other system parameters, but (22) is probably more useful because t_c is easier to measure than the flow rate Q. When t_c is large, (22) resembles Rayleigh's formula (1) with a numerical factor π instead of 3.8. Rayleigh's coefficient 3.8 was obtained from experiments with water and in fact varied somewhat as a function of the dimensionless group $\gamma/\rho g R_0^2$. In his case the splitting of the pendant drop would be dominated by inertia, rather than viscosity, so that the present theory would not apply.

Finally we consider the periodic dripping which is eventually established. Suppose we set $t^* = 0$ at the instant when a drop breaks away. At this instant, then, the fluid element just emerging is labelled $\tau^* = 0$ and the fluid element at the bottom of the remaining thread is labelled $\tau^* = -\tau_0^*$, which is to be determined. This thread will recoil somewhat under the action of surface tension. However this does not matter and we only have to replace (13) by

$$SA+2\pi R\gamma = \rho g Q(\tau+\tau_0), \tag{23}$$

since only the volume matters, not the shape. The analysis now proceeds almost exactly as above, with different letters, and there results, instead of (18),

$$t^* = \tau^* - \frac{2}{\delta}\left[1 + \frac{\tau^*+\tau_0^*}{3\delta}\log\left(1 - \frac{3\delta}{\tau^*+\tau_0^*}\right)\right], \tag{24}$$

and instead of (19),

$$(\tau_c^*+\tau_0^*)^2 = \frac{6}{\lambda'}\left[\frac{1}{\lambda'}\log(1-\lambda') + \frac{1}{1-\lambda'}\right], \tag{25}$$

with

$$\lambda' = \frac{3\delta}{\tau_c^*+\tau_0^*}. \tag{26}$$

The volume breaking away is essentially $\tau_0^*+\tau_c^*$ and in view of (25) and (26) this is still given as a function of δ by the curve shown in figure 2, regardless of the actual value of τ_0^*. This is determined by a periodicity condition as follows.

Let t_c^* be the time at which the new drop breaks away, and recall that it does so on the element τ_c^*. The volume left behind is $t_c^*-\tau_c^*$; this is the difference between the τ^*-values of the fluid element at the break point, τ_c^*, and the fluid element just emerging from the tube, t_c^*. This volume must equal τ_0^*. Hence

$$t_c^* - \tau_c^* = \tau_0^*,$$

or

$$t_c^* = \tau_c^* + \tau_0^*. \tag{27}$$

Equation (27) can also, of course, be interpreted as stating that the volume extruded during the time interval between two drops must equal the drop volume. This, as noted, is already determined as a function of δ, so τ_0^* can be determined from (24), if desired.

The quantity of practical interest is the drop size, which is essentially t_c^*. A useful formula can be obtained by noting that the curve of t_c^* against δ is almost straight, up to $\delta \approx 1$, so that

$$t_c^* \approx \surd 3 + 2.29\delta. \tag{28}$$

The dimensional form of this can be written

$$\Delta m = \left(\frac{3\mu A_0 \rho Q}{g}\right)^{\frac{1}{2}} + 2.29\frac{\pi}{3}\frac{\gamma R_0}{g}$$

$$= \left(\frac{3\mu A_0 \rho Q}{g}\right)^{\frac{1}{2}} + 2.40\frac{\gamma R_0}{g}, \tag{29}$$

and it is this formula that has been tested by experiment.

3. Experiment

The apparatus, which was constructed from Perspex, is sketched in figure 3. A vertical cylinder, about 25 cm × 2 cm diameter, is closed by a disk with a hole drilled through it, which in turn carries a removable exit pipe. The lower end of this was tapered by machining, so as to give a reasonably sharp rim to the orifice. In the experiments reported here, the orifice diameter was 3.5 mm, and the exit pipe was about 5 cm long.

The working fluid was Lyle's Golden Syrup used as purchased. The viscosity was measured by the falling-ball method and found to be 62.5 $\mathrm{Ns/m^2}$ at 20 °C. No method

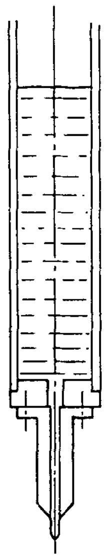

FIGURE 3. Sketch of experimental arrangement.

of measuring surface tension was available and the value 80×10^{-3} kg s^{-2} was taken from the literature (Sinat-Radchenko 1982).

No attempt was made to control the flow rate directly; the fluid was poured into the cylinder to various levels and simply allowed to flow out under its own weight.

Once a steady dripping was established a group of drops, usually 5, was timed with a stop watch, and collected and weighed. This procedure was carried out 3 or 4 times as quickly as possible, the idea being that these would represent measurements at a constant flow rate. Then the level of fluid in the cylinder was increased by 1–2 cm and another set of measurements made.

The time intervals between the drops remained very nearly constant; the times taken for each group of 5 drops (usually in the range 40 s to 2 min, i.e. 8–25 s per drop) varied by less than 1 % at each fixed flow rate. The mass collected was also very reproducible with variation from group to group at fixed flow rate being less than 1 %. Typical drop masses were 75–85 mg; the balance used had a nominal accuracy of 0.1 mg.

There was no way to control the temperature of the apparatus and the temperature in the laboratory rose by about 1 °C during the course of the experiments summarized in figure 4. Thus the later experiments, those at higher flow rates, were done at a somewhat lower viscosity and the data points are probably about 5 % too low.

Comparing experiment with theory (figure 4) we see that the theoretical curve gives the right trend and the right order of magnitude but falls consistently below the experimental points by about 25 %, which suggests a reasonably important systematic error. In fact several were tracked down.

(i) Although the nozzle appeared sharp to the naked eye it was found on examination under a microscope to have a rather ragged rim whose thickness was estimated at 0.13 mm. This is a limitation of Perspex. This rim would certainly be wetted and the effective radius would be increased by about 7 %. It is also possible

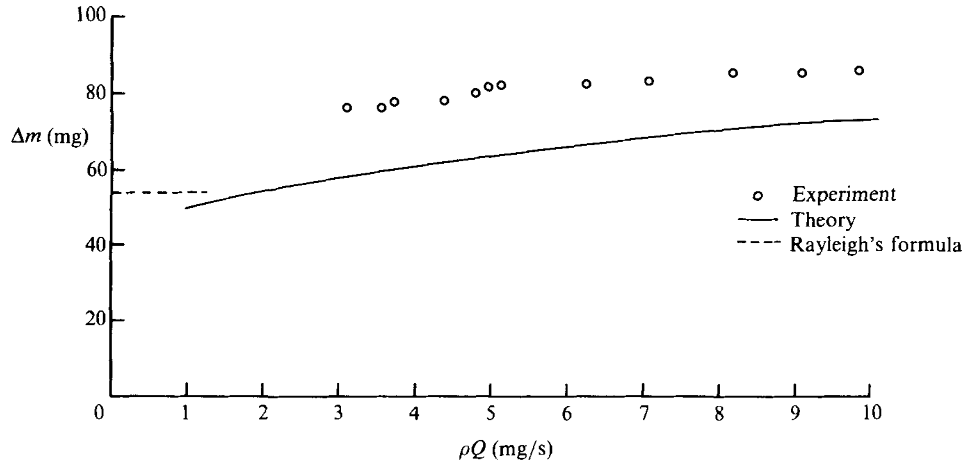

FIGURE 4. Comparison of theory and experiment.

that the fluid wetted a portion of the outer surface of the nozzle; however this was not apparent on inspection with a hand lens.

(ii) The flow rate increases slightly as the drop forms. This is because the fluid is pulled through the exit pipe by the drop as well as pushed through by the fluid above. This effect will be more important when the fluid level in the cylinder is low, when it can be estimated that the flow rate increased by about 10%. The analysis can be modified to allow for variable flow rate. After lengthy calculations, which will not be given here, it turns out that if the flow rate increases during drop formation, as it does, the theoretical estimate of drop size using the mean flow rate (equation (29)) is an underestimate. Assuming that the flow rate increases linearly (which is not quite correct) by a fraction β during the formation of each drop, then the theoretical estimate should be revised upwards by a fraction $\sqrt{3}\beta/4$ or about 4% maximum in our case.

(iii) More important probably than either of these is the swelling of the jet as it emerges from the nozzle. Die-swell is well known in connection with the extrusion of elastic liquids but occurs also with Newtonian liquids, although this seems to be less well known. For creeping flow in the absence of gravity and surface tension a swell ratio of about 13% seems to be accepted and the experiments of Goren & Wronski (1966) and the finite-element calculations of Nickell, Tanner & Caswell (1974) may be cited. Gravity was present, of course, in the experimental work just cited and efforts have been made to include it in the numerical simulations. Calculations have been published by Dutta & Ryan (1982) for example, but objections have been raised (Vrentas, Vrentas & Shirazi 1985) to the form of downstream conditions used. All this work refers to steady jets of course and is not directly applicable. But it seems likely that the emerging thread will expand by about 10% or so. The absence of a good estimate of the swell ratio is the major weakness in the present theory.

Taking all these corrections into account, it seems that the theoretical curve should be revised upwards by about 15–20% and the agreement can then be regarded as satisfactory. A formula which is reasonably accurate numerically can thus be obtained by using (29) with a nominal radius 20% greater than the true radius.

4. Concluding remarks

The agreement between theory and experiment suggests that the model has the correct mechanism of drop elongation and rupture, and that the major source of the discrepancy is the failure of the one-dimensional theory near the nozzle exit. It is difficult to see how this theoretical deficiency can be remedied.

An attempt was made to test (22) experimentally but then the measured values of t_c showed considerable scatter and the general agreement between theory and experiment was slightly worse than for the periodic dripping process. There was no reliable way to start the flow off (I simply put a finger over the hole and then removed it). The drop grows very slowly at first and much of the time up to $t = t_c$ is spent in a flow regime in which the one-dimensional theory must fail. In typical experiments the drop was still only about one tube radius long at $t = \frac{1}{2}t_c$. Some allowance could be made for this, but added to all other corrections this would give a combined correction factor of the same order as the basic theoretical result, and this does not seem worthwhile.

In the course of setting up the experiment some other observations were made which may be of interest. An attempt was made to use a silicone fluid (PDMS) of similar viscosity but which had a much lower value of the surface tension (about 21×10^{-3} kg s^{-2}). Although drops formed they were reluctant to separate completely and instead pulled out a long fine fluid thread behind them, which would often persist until the next drop fell. The thread could be ruptured by allowing the drop to fall a long way, but then the fluid thread tended to blow about causing errors in the drop weight measurements. This suggests that surface tension may play a role in thread rupture other than what is allowed for in the present theory. Note also that the extension rate of the thread is limited by the inertia of the falling drop, which cannot attain an acceleration greater than g. The strain rate and therefore the stress will in fact rise to a maximum and then fall to zero algebraically as $t \to \infty$, rather than tend to infinity at a finite time. One might conjecture then, that the threads generally break because of some instability driven by surface tension, which in the case of PDMS does not have time to operate before thc strain rate begins to decrease.

The instability and break-up of extending fluid threads is of great interest in connection with the spinning of polymer melts and solutions (White & Ide 1975; Bousfield *et al.* 1986). The conclusion seems to be that elasticity may have a long-term stabilizing effect. It was initially hoped that the present experiment might provide some information on this topic, offering as it does a whole family of constant-force stretching experiments for each drop; and it is intended to attempt to measure the thread length $X(\tau_0, t)$. However the uncertainty about conditions at the pipe exit causes serious theoretical difficulties.

When the flow rate is large enough a steady jet will form but the process seems to depend somewhat on how the flow is increased. For a large orifice (about 4.4 mm diameter) the silicone fluid formed a thin jet even at very low flow rates and could hardly be induced to form drops at all. For fluids of lower viscosity (1–10 Ns/m^2) the fluid often dripped at first but developed into a steady jet later. Each drop formed before its predecesor had fallen away completely and the interval between successive drops gradually got less. (Usually this resulted in a steady jet, as noted, but occasionally a periodic blobbing motion could be set up in which the fluid formed a continuous thread). This behaviour might be the result of each drop increasing the flow rate (by the mechanism discussed earlier) and thereby increasing the size of its successor, until there is no longer time for them to separate before they hit the bottom of the collecting vessel.

Much preliminary experimentation was carried out while the author was visiting the Department of Aerospace Engineering and Mechanics, University of Minnesota, at the invitation of Professor D. D. Joseph. It is a pleasure to thank Professor Joseph for his hospitality and support. The final experiments were done in the Rheology Laboratory of the Mathematics Department, UMIST, and the author would like to express his thanks to Dr A. Kaye for this use of this facility.

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